Applications of Partial Derivatives II

Partial Elasticities

Let $q_x = f(p_1, p_2)$ be the demand for commodity X, which depends upon the prices

p_1 and p_2 of commodities X and Y respectively. Similarly demand for commodity Y which also depends on p_1 and p_2 can be given by another function.

$q_y = g(p_1, p_2)$

If we have two commodities then we can calculate, 2 direct partial elasticities and 2 cross partial elasticities.

The direct partial elasticity of demand q_x with respect to p_1 is defined as

$$E_{11} = (P_1/ Q_1).(\partial Q1/\partial P_1) \quad (p_2 \text{ is constant})$$

Likewise, we can write the direct partial elasticity of quantity demanded of Y w.r.t. p_2 (price of Y) where we assume p_1 is constant.

The **cross partial elasticity** of demand commodity x with respect to price of commodity measures the responsiveness of demand of X to change in price of Y keeping price of X constant. Mathematically it is written as

$$E_{12} = (P_2 / Q_1). (\partial Q1/\partial P2) \qquad (p_1 \text{ is constant})$$

Similarly, we can find relation between quantity demanded of Y w.r.t price of X

$$E_{21} = (P_1 / Q_2).(\partial Q2/\partial P1) \qquad (p_2 \text{ is constant})$$

Sign of cross elasticities tells us about the relationship between two commodities. The coefficient of cross elasticity can be zero, positive or negative. Cross elasticity of two unrelated commodities shall be zero for example car and TV.

Cross elasticity of demand for substitute (or competitive) goods is always positive i.e. demand for a good increases when the price for its substitute increases. A simple example of pair of substitute goods is tea and coffee. If all factors are constant then an increase in price of tea is expected to lead to decrease in demand for tea and increase in demand for coffee. If price of x increases and demand for good Y also increases the derivative will be positive in the above formula.

Companies utilize cross-elasticity of demand to establish prices to sell their goods. Incremental price changes to goods with substitutes are analyzed to determine the appropriate level of demand desired and the associated price of a substitute. An increase in price of a brand of toothpaste can lead to an increase in demand for a competitor brand. Products with no substitutes can have the ability to be sold at higher prices because there is no cross-elasticity of demand to consider.

Alternatively, the cross elasticity of demand for complementary goods is negative. As the price of a commodity increases, the demand for its complementary good decreases. Most simple examples of a such a pair of commodities is pen and ink or car and petrol. An increase in price of pen leads a fall in demand for pens leading to a decrease in demand for ink. Thus, there is a negative relation between price of pen and demand for ink.

Thus

$\partial Q1/\partial P2 > 0$ and $\partial Q2 /\partial P1 > 0$ ………. competitive goods

$\partial Q1/\partial P2 < 0$ and $\partial Q2 /\partial P1 < 0$ ………. complementary goods

$\partial Q1\partial/P2 = 0$ and $\partial Q2 /\partial P1 = 0$ ………unrelated goods

It is important to note that partial elasticity is a theoretical concept where we are assuming **all other** factors affecting demand are constant.

Example:

1. $Q_1 = 200 - 5P_1 + 4P_2$ (demand function for commodity X)

   $Q_2 = 300 + 2P_1 - 4P_2$ (demand function for commodity X)

   Calculate cross elasticities and determine the relationship between X and Y given $P_{1=}$ 20 and $P_2 = 20$

   $\partial Q_1/\partial P_2 = 4$

   $\partial Q_2 /\partial P_1 = 2$

At $P_{1=}$ 20 and $P_2 = 20$, $Q_1 = 120$ and $Q_2 = 260$. Substitute these values we find $E_{12} = (P_2 / Q_1).(\partial Q1/\partial P2) = 2/3$ and $E_{21} = (P_1 / Q_2).(\partial Q2/\partial P1) = 2.6$
Both cross elasticities are positive therefore they are competitive goods.

2. The monthly demand equations for the sale of cricket bats and balls in a sporting goods store are
   $Q1 = 500 − 0.5P_1 − P_2{}^2$ (Bats)
   $Q2 = 10,000 − 8P_1{}^2 − 100P_2$ (Balls),
   determine whether the indicated products are competitive, complementary, or neither. Find cross elasticities $P_1 = 100$ and P_2 is 20

   Answer : complementary

3. Suppose the demand for good X is given by Qx=10-3Px+2Py-4Pz-1M, where Px is the price of good X, Py is the price of good Y, Pz is the price of good Z, and M is income. What is the relation between X and Y. What kind of goods are X and Z ?
   (hint : treat M as a constant)

4. In Q No 3 what is the income elasticity of demand for commodity X ?

Applications of Total Derivative

The total differential of a function z = f(x, y) is defined as

The quantity

$$dz = \frac{\partial z}{\partial x}\Delta x + \frac{\partial z}{\partial y}\Delta y$$

is called the **total differential** of the function z = f(x, y). If the increment is infinitesimally small, then total differential is written as

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy\ .$$

It represents total change in Z which is a sum of change in Z due to X **and** change in Z due to Y

$Z = 20X^3 + 3Y – 2XY$

$dZ = (60\ X^2 - 2Y)\ dX + (3\text{-}2X)\ dY$

Isoquants :

Assume a scenario where there are only 2 factors of production L and K and they are substitutable. As you increase L, you quantity of capital used will be decreased to produce the same level of output. Thus there can be various combinations of labour and capital which can that produce same level of output.

$Q = f(L,K)$

An **isoquant** is a curve that shows all the combinations of inputs that produce same level of output. 'Iso' means equal and 'quant' means quantity. (It is a concept similar to Indifference curve in consumer theory. Variuos combinations of X and Y consumed to attain same level of satisfaction)

Therefore, an **isoquant** represents a constant quantity of output produced by different combinations L and K in our case above. If we plot these different combinations, we get a curve called Isoproduct curve or Isoquant curve. Along the curve level of Q is constant. In other words, change in Q along the curve is zero. Consider the following graph. Here, L and K are measured on X and Y axis respectively. At point A and B, same output of 10 (Rs. 000) can be produced using different combinations of L and K. Higher isoquant represents higher output level.

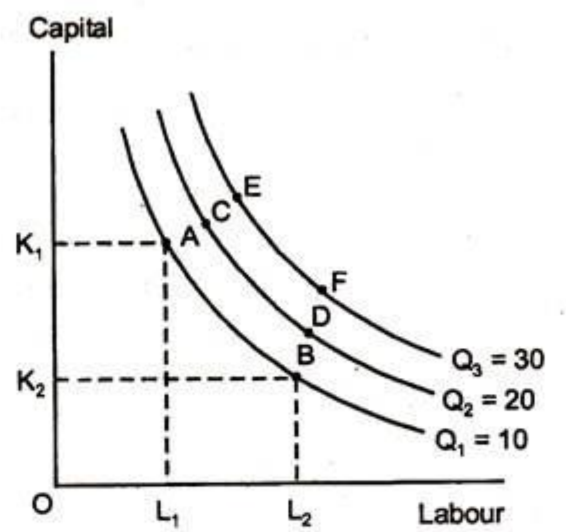

Fig. 6.3 : Isoquant Curve/Isoquant Map

Diagram source:http://www.economicsdiscussion.net/notes/study-notes-on-isoquants-with-diagram/16342

$Q = f(L,K)$

Differentiating both sides and finding total derivative of Q we get α

$dQ = (MP_L)\ dL + MP_k\ .\ dK$

Total change in output = change in output due to change in L(keeping K constant) + change in output due to change in K (keeping L constant)

Along an isoquant dQ= 0. (As output remains constant along the curve, change in output is zero)

$dQ = (MP_L)\ dL + MP_k\ .\ dK = 0$

$dK/dL = -(MP_L / MP_k)$ This is slope of an isoquant which measures the rate at which capital can replace per unit of labour, keeping output constant. This slope is called the marginal rate of technical substitution of capital for labour (MRTS).

Consider Cobb Douglas production function $Q = AL^a K^\beta$

$dQ = (MP_L)\ dL + MP_k\ .\ dK$

$= A\alpha L^{\alpha-1} K^\beta\ dL + A\beta\ L^a\ K^{\beta-1} dK = 0$

$dK/dL = -(\alpha L^{\alpha-1} K^\beta)/(\beta\ L^a\ K^{\beta-1}) = -\alpha K/\beta L$

$= -(MP_L / MP_k) = MRTS$

Additional source: Refer https://www.youtube.com/watch?v=y_fwnYev2yo